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Δ - Monotone Arrangements of Real Numbers.

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Δ-Monotone Arrangements of Real Numbers*

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Summary

Is it possible to arrange a given sequence of MN real numbers into an $M \times N$ matrix such that all second differences as defined by (2) are non-negative? The answer is affirmative for $M = 2$ and arbitrary N, and also for $M = N = 3$. In these cases there is a uniform rule, valid for all sequences, stating for each i the position in the matrix assigned to the i-th smallest number of the given sequence. For $M = 3$ and $N = 4$ the answer is again affirmative, but for this and larger matrices no such uniform rule is valid for all sequences simultaneously. The problem for larger M and N is open.

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Given a set of N arbitrary real numbers, it is always possible to label them a_i, $i = 1,2,\ldots,N$, such that

(1) $$\Delta_i = a_{i+1} - a_i \geq 0 \;, \qquad i = 1,\ldots,N-1.$$

An attempt to generalize this statement for two dimensions raises the following question. Given a set of MN arbitrary real numbers, is it always possible to label them as a_{ij}, $i = 1,\ldots,M$; $j = 1,\ldots,N$, such that

(2) $$\Delta_{ij} = a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{ij} \geq 0 \;;$$
$$\text{for } i = 1,\ldots,M-1; \; j = 1,\ldots,N-1.$$

The problem rephrased in matrix notation becomes: given a set of MN arbitrary real numbers, is it possible to arrange them into an $M \times N$ matrix such that for every 2×2 submatrix, the sum of the numbers on the main diagonal is larger than or equal to the sum of the entries on the other diagonal? A matrix having this property may be called a Δ-monotone matrix.

From now on we reserve the name "square" for a 2×2 matrix consisting of four neighbor elements $a_{ij}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}$. On first sight Δ-monotonicity seems stronger than (2), as it refers to all 2×2 submatrices and not only to the squares. But it is obvious that a second difference in any 2×2 submatrix can be written as the sum of such differences in the squares of which it consists.

A sufficient, but not necessary, condition for (2) is that $a_{ij} \geq a_{i+1,j}$ and $a_{i+1,j+1} \geq a_{i,j+1}$. If we place arrows originating in the larger number and pointing to the smaller, the configuration of Fig. 1 (called a vertical arrangement, V) ensures a nonnegative difference. The same holds for Fig. 2 (horizontal arrangement, H).

$$\begin{tikzcd} a_{ij} \arrow[d] \arrow[dr, phantom, "V"] & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1}. \arrow[u] \end{tikzcd}$$

Fig. 1

$$\begin{tikzcd} a_{ij} \arrow[r] \arrow[dr, phantom, "H"] & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \arrow[l] \end{tikzcd}$$

Fig. 2

Let the given MN numbers be arranged in a non-decreasing sequence

(3) $$b_1 \leq b_2 \leq \cdots \leq b_{MN} .$$

Then the case M = 2, N arbitrary can be solved by a sequence of horizontal arrangements (Fig. 3) and the case M = 3, N = 3 by two horizontal and two vertical, ones (Fig. 4).

$$\begin{array}{ccccccccc}
b_{2N} & \longrightarrow & b_{2N-1} & \longrightarrow & b_{2N-2} & \cdots & b_{N+2} & \longrightarrow & b_{N+1} \\
 & H & & H & & H & & H & \\
b_1 & \longleftarrow & b_2 & \longleftarrow & b_3 & \cdots & b_{N-1} & \longleftarrow & b_N
\end{array}$$

Fig. 3

$$\begin{tikzcd}
b_9 \arrow[r] \arrow[d] \arrow[dr, phantom, "V"] & b_4 \arrow[r] \arrow[dr, phantom, "H"] & b_2 \\
b_6 \arrow[r] \arrow[d] \arrow[dr, phantom, "H"] & b_5 \arrow[u] \arrow[d] \arrow[dr, phantom, "V"] & b_7 \arrow[l] \arrow[u] \\
b_1 & b_3 \arrow[l] & b_8 \arrow[l] \arrow[u]
\end{tikzcd}$$

Fig. 4

The arrangements displayed in Figures 3 and 4 are based exclusively on the indices of the ordered numbers as given by (3): there is a 1:1 function $F(m) = (j,k)$ mapping the indices $m = 1,\ldots,MN$ onto the pairs (j,k) for $j = 1,\ldots,M$ and $k = 1,\ldots,N$. For example in Figure 4, $F(9) = (1,1)$, $F(4) = (1,2)$ etc. Such an arrangement of MN numbers into a Δ-monotone matrix will be called a "uniform" solution implying thereby that it holds irrespective of the magnitudes of the numbers.

An arrangement of the numbers will be said to have "circularity" or a "circular path" if it is possible to start from a corner of a square and arrive at the same place after following a path directed by the arrows.

The following theorems show that the arrow device has only limited value and that a uniform solution is impossible for $M \geq 3$ and $N \geq 4$.

Theorem 1. A uniform solution cannot have circularity anywhere.

Proof: Circularity implies that all the numbers in that path are equal which in turn implies that the solution is not uniform.

Theorem 2. Every square in a uniform solution has to have either horizontal or vertical arrangement.

Proof: Suppose one of the squares is neither circular nor has the horizontal or vertical arrangement.
For example consider

$$\begin{tikzcd}
(i,j) & (i,j+1) \arrow[l] \\
(i+1,j) \arrow[u] & (i+1,j+1) \arrow[l] \arrow[u]
\end{tikzcd}$$

Fig. 5

Further, suppose that the solution written in the functional form is such that $F(\ell) = (i,j)$, where $b_1 \leq \cdots \leq b_\ell \leq b_{\ell+1} \leq \cdots \leq b_{MN}$. Obviously the indices which are mapped into $(i,j+1)$, $(i+1,j)$ and $(i,j+1)$ are larger than ℓ. Now, if the set of MN numbers is chosen to be $b_1 = b_2 = \dots = b_\ell = 0$ and $b_{\ell+1} = \dots = b_{MN} = 1$; the square under consideration destroys Δ monotonicity. The same argument holds for the other squares which do not have vertical or horizontal arrangements.

Theorem 3. A uniform solution can have at most one vertical arrangement in a row and at most one horizontal arrangement in a column. Thus a uniform solution can have at most M-1 vertical and N-1 horizontal arrangements.

Proof: Suppose there are two squares with the vertical arrangement in the same row. Obviously these cannot be adjacent. Suppose they are separated by a chain of squares with the horizontal arrangement. This, however, leads to circularity (see Figures 6a and 6b) which is not admissible in a uniform solution by Theorem 1. This shows that the assertion regarding the vertical arrangements holds. That for the horizontal arrangements follows in the same manner.

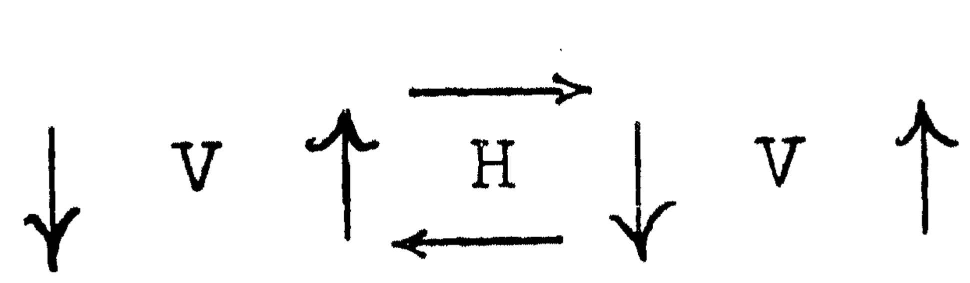

Fig. 6a

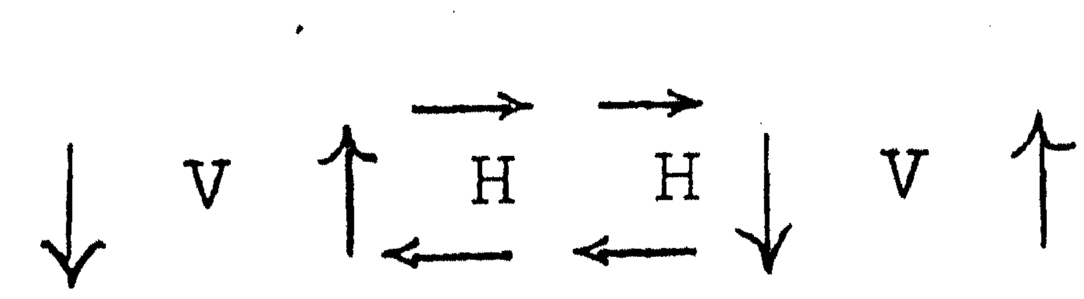

Fig. 6b

Theorem 4. A uniform solution for $M \geq 3$ and $N \geq 4$ (or for $M \geq 4$, $N \geq 3$) does not exist.

Proof: There are (M-1)(N-1) squares and by Theorem 3 at most M+N-2 of these can have the vertical or the horizontal arrangements. The nonexistence of a uniform solution follows from Theorem 2.

Above theorems of course do not imply that Δ monotone arrangements are not possible.

However it seems reasonable to suppose that it will be increasingly difficult to find a Δ-monotone arrangement as M and N increase. The absence of a uniform rule shown in Theorem 4 does not mean however that it is a hopeless task.

Theorem 5. Any sequence of 12 real numbers can be arranged into a Δ-monotone 3×4 matrix.

Proof: In Figures 7, 8, 9 all squares have the horizontal (H) or the vertical (V) arrangement except for those marked *. It follows that the arrangement in Fig. 7 works as soon as $b_8 - b_7 \geq b_4 - b_3$. Similarly, the rule of Fig. 8 works as soon as $b_{12} - b_{11} \geq b_8 - b_7$. In the remaining case $(b_4 - b_3 > b_8 - b_7 > b_{12} - b_{11})$ the rule of Fig. 9 works.

$$\begin{array}{ccccccc} b_{12} & & b_{11} & & b_{10} & & b_5 \\ & H & & H & & V & \\ b_2 & & b_3 & & b_7 & & b_6 \\ & V & & * & & H & \\ b_1 & & b_4 & & b_8 & & b_9 \end{array}$$

Fig. 7

$$\begin{array}{ccccccc} b_{10} & & b_3 & & b_2 & & b_1 \\ & V & & H & & H & \\ b_9 & & b_6 & & b_7 & & b_{11} \\ & H & & V & & * & \\ b_4 & & b_5 & & b_8 & & b_{12} \end{array}$$

Fig. 8

$$\begin{array}{ccccccc} b_{10} & & b_5 & & b_4 & & b_3 \\ & V & & H & & * & \\ b_9 & & b_6 & & b_8 & & b_7 \\ & H & & V & & * & \\ b_1 & & b_2 & & b_{12} & & b_{11} \end{array}$$

Fig. 9

The problem for larger M and N is still open. So far no counterexample for $M = N = 4$ has been found, and it seems probable that the proof of Theorem 5 could be extended to cover this case. However, each figure now leads to three inequalities as only 6 of the 9 squares can be made H or V (Theorem 3).

As a passing remark it should be noted that the possibility of Δ monotone arrangement remains invariant under the shift and scale transformations of the given sequence of numbers. Also a convex combination of two Δ monotone matrices is Δ monotone.